AD-A091 522 CARNEGIE-MELLON UNIV PITTSBURGH PA MANAGEMENT SCIENC--ETC F/G 12/1
PROFIT MAXIMIZATION MODELS FOR EXPONENTIAL DECAY PROCESSES.(U)
AUG 80 S P SETHI, G L THOMPSON, V UDAYABHANU N00014-75-C-0621
UNCLASSIFIED MSRR-464 NL

1 of 1
AD
A091522

END
DATE
FILMED
12 80
DTIC

LEVEL II

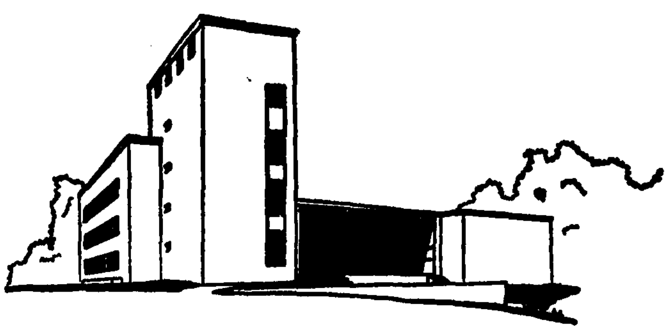
AD A091522

Carnegie-Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

WILLIAM LARIMER MELLON, FOUNDER



80 11 04 019

W.P. # 9-80-81

(1)

(9) Management Sciences Research Report No. 464

(6) PROFIT MAXIMIZATION MODELS FOR

EXPONENTIAL DECAY PROCESSES

by

(10) S. P. Sethi*
G. L. Thompson**
V. Udayabhanu**

(11) August, 1980

(14) MSRR-464, WP-9-80-81

* University of Toronto
** Carnegie-Mellon University

(12) 32

This report was prepared as part of the activities of the Management ~~Sciences Research~~ Group, Carnegie-Mellon University, under Contract (15) N00014-75-C-0621 NR 047-048 with the U. S. Office of Naval Research, ~~and Grant A4619 from~~ NSERC, Canada. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

Management Sciences Research Group
Graduate School of Industrial Administration
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

403426

PROFIT MAXIMIZATION MODELS FOR

EXPONENTIAL DECAY PROCESSES

by

S.P. Sethi, G.L. Thompson and V. Udayabhanu

ABSTRACT

A number of real world processes can be modelled as exponential decay processes. Examples are: machine replacement, oil well extraction, advertising goodwill, repair and cleaning activities, etc.. In this paper we analyze a series of discounted or undiscounted, deterministic or stochastic exponential decay models. We characterize finite and infinite horizon optimal solutions for each model. We show that the solution can be characterized for the oil drillers problem in the following way: Once a well of sufficient capacity is drilled, oil is pumped from it until the oil remaining decreases to a fixed cut-off level; then the well is abandoned, and a new well is drilled. The resulting process when repeated over time appears to be the same as an oil source which produces oil revenue continuously at the fixed cut-off level. In other words, the excess revenue received from an oil well when its capacity is greater than the cutoff level is just sufficient to pay for drilling costs for a new well.

Key Words

Exponential decay processes
Oil driller's problem

1. Introduction

Several real-world processes are subject to the phenomenon of exponential decay. Here we analyze various optimization models involving such processes.

Consider, as a first example, a machine replacement problem. The performance level of the machine - which may be measured in terms of the output quantity and the production rate - can be assumed to decrease exponentially with time. The machine earns revenue during its lifetime, and then must be replaced at a certain cost. The question, therefore, is to decide on the timing of machine replacement, and the capacity (in terms of initial performance level) of the new machine [2, 6, 7].

Another example concerns the determination of the optimal drilling times and capacities of oil wells [7]. The rate of extraction of oil from the well may be assumed to be directly proportional to the quantity of oil remaining in the well, which, in turn, implies that the process is subject to exponential decay. Profits are generated during the course of oil extraction; at some point, the well is abandoned, and a new one drilled. The same model is representative of mining operations in general.

Memory is yet another process subject to exponential decay; consequently, our models may be used to formulate an advertising policy problem [4, 5]. Media commercials are aimed at generating goodwill in the minds of consumers, leading to enhanced sales. The loss of goodwill, as a consequence of the decline in memory, has to be countered by exhibiting further commercials; the timing and quantity of commercials become the decision variables.

The periodic repainting of inns [3], and the cleaning of swimming pools, which provide recreational benefits to society at large, are further examples of situations which fit into the general framework of this paper.

Section 2 lays down the preliminary groundwork for the models; Sections 3, 4, and 5 analyze the deterministic discounted, deterministic undiscounted, and stochastic models respectively.

2. The Oil Driller's Problem

We now present several models involving exponential decay, framed in terms of the oil driller's problem. We assume that at any instant, there is at most one functioning oil well. The rate of oil extraction is given to be a constant D times the quantity of oil $C(t)$ remaining in the well at time t. The oil is sold at a unit profit of P. As the oil reserves get depleted, the rate of extraction eventually decreases to uneconomic levels, making it worthwhile to abandon the well and drill a new one at a cost of $f(v)$, where v is the capacity of the well. The function $f(v)$ is assumed to be continuous and twice differentiable. We may further suppose, without loss of realism, that $v \geq 0$, $f(0) = 0$, and $f'(v) > 0$.

The rate of oil extraction, then, is represented by

$$\dot{C}(t) = -DC(t). \tag{1}$$

Using the boundary condition

$$C(0) = v, \tag{2}$$

the solution to (1) may be obtained as

$$C(t) = ve^{-Dt}, \tag{3}$$

where time is measured from the instant at which the well is drilled. Drilling is assumed to be instantaneous.

To complete the model, we specify the objective function:

$$\underset{v_i,t_i}{\text{Maximize}}\ J = \sum_i \left[\int_{t_i}^{t_{i+1}} PDv_i e^{-D(t-t_i)} dt - f(v_i)\right] . \tag{4}$$

The drilling times t_1, $t_2, \ldots,$ and the corresponding well capacities $v_1, v_2, \ldots,$ constitute the decision variables.

3. Deterministic Discounted Models

In this section, we shall analyze a deterministic infinite horizon model, with a continuous discount rate $r > 0$.

It is clear by symmetry that the interval T between drillings will remain constant; so will the capacity v of the oil well; see Figure 1. Note its

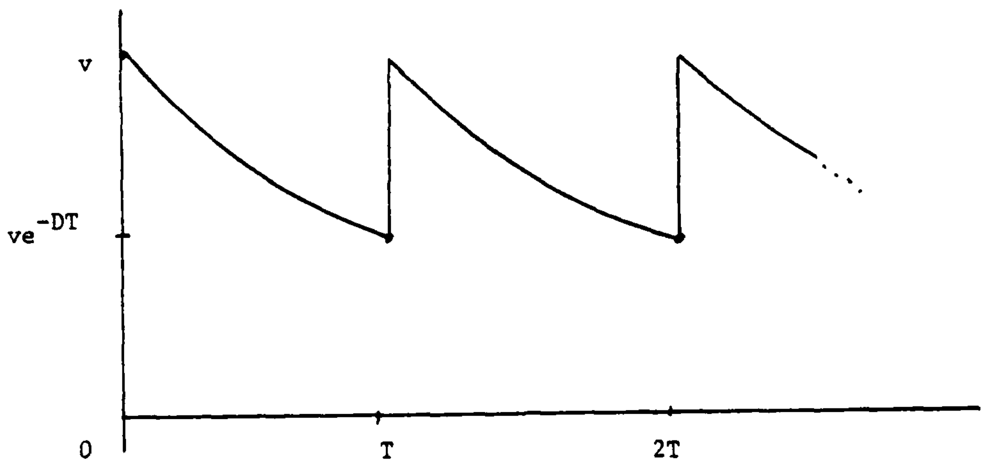


Figure 1. Oil well capacity over time.

similarity to the simple EOQ lot size model. If J represents the infinite horizon profit function, we may write

$$J = \int_0^T PDve^{-Dt}e^{-rt}dt - f(v) + Je^{-rT}, \tag{5}$$

which may be more conveniently expressed as

$$J = \frac{v}{(1-e^{-rT})}\left[\frac{PD(1-e^{-(D+r)T})}{(D+r)} - \frac{f(v)}{v}\right] . \tag{6}$$

The objective is to maximize J with respect to v and T. Further analysis depends on the form of the cost function; we consider two cases.

<u>Case 1:</u> $f''(v) \le 0$.

Here $f(v)$ is concave in v, so that $\frac{f(v)}{v}$ is nonincreasing with respect to v. A cursory examination of (6) makes it patent that J is maximized by letting v become as large as possible. Hence we need an upper bound on the drilling capacity; we assume that $v \le V < \infty$.

For the reason just mentioned, the optimal drilling capacity must be at the upper bound, provided that it yields a positive profit. Thus we may write

$$v^* = \begin{cases} V & \text{if } PDV(1-e^{-(D+r)T^*}) > (D+r)f(V) \\ 0 & \text{otherwise} \end{cases} . \tag{7}$$

The optimal drilling interval is obtained by differentiating J with respect to T and equating it to zero; hence T^* is determined from the relation

$$(D+r)e^{-DT^*} - De^{-(D+r)T^*} = r\left[1 - \frac{(D+r)}{PD} \cdot \frac{f(V)}{V}\right] , \tag{8}$$

which is valid under the hypothesis that $v^* = V$. Any solution to (8) yielding $T^* \le 0$ implies that $v^* = 0$; further, $T^* = \infty$ only if $PDV = (D+r)f(V)$, in which case, we must have $v^* = 0$, from (7).

Case 2: $f''(v) > 0$.

Here $f(v)$ is strictly convex in v, so that $\frac{f(v)}{v}$ is an increasing function of v. This property of decreasing returns to scale eliminates the need for an upper bound on v so that v^* will always be finite.

Equating to zero the derivatives of J with respect to v and T, the optimal values v^* and T^* are obtained as the simultaneous solutions to the equations

$$f'(v^*) = \frac{PD(1 - e^{-(D+r)T^*})}{(D+r)} \tag{9}$$

and

$$\frac{e^{rT^*} - 1}{e^{(D+r)T^*} - 1} = \frac{r}{(D+r)} \left[1 - \frac{f(v^*)}{v^* f'(v^*)}\right] . \tag{10}$$

We now show that it is always optimal to drill a positive quantity.

Proposition 1: $v^* > 0$.

Proof: Using (9), the profit function (6) may be evaluated at the optimum as

$$J^* = \frac{1}{1-e^{-rT^*}} \left[v^* f'(v^*) - f(v^*)\right]. \tag{11}$$

Now strict convexity of $f(v)$ implies that $f'(v) > \frac{f(v)}{v}$; further, since $f'(v) > 0$ by assumption, it may be deduced from (9) that $T^* > 0$. Consequently, (11) implies that $J^* > 0$, from which we may infer that $v^* > 0$.

Q.E.D.

Equations (9) and (10) may be employed to further simplify (11), yielding

$$J^* = \frac{PDv^* e^{-DT^*}}{r} . \tag{12}$$

It is easy to infer from (12) that $J^* < \infty$ and $T^* < \infty$. Thus v^*, T^*, and J^* are strictly positive and finite-valued. In Figure 2 we have sketched the optimal solution. Equation (12) affords an interesting interpretation.

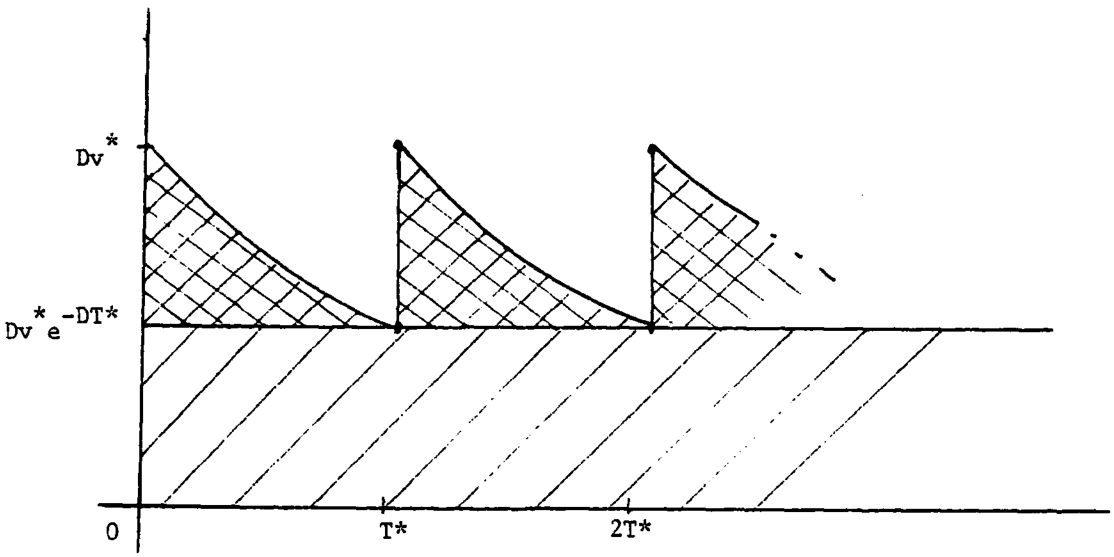


Figure 2. Optimal oil extraction rate over time.

Notice that

$$\int_0^\infty PDv^*e^{-DT^*}e^{-rt}dt = \frac{PDv^*e^{-DT^*}}{r} .$$

Thus J^* is the infinite horizon discounted profit which would be obtained by selling oil at the constant rate of $Dv^*e^{-DT^*}$, with no drilling costs; further, this constant rate is precisely the rate of oil extraction just at the instant before the well is abandoned in favor of a new one. In Figure 2, J^* is the discounted value of the shaded area. Therefore the discounted value of the cross-hatched area equals the discounted value of the drilling costs.

<u>Example</u>:

Suppose $D = 1$, $r = 0.1$, $P = 100$

<u>Case 1</u>: $f(v) = 50v, \quad V = 40.$

Equation (8) reduces to

$$1.1e^{-T^*} - e^{-1.1T^*} = 0.045,$$

yielding $T^* = 1.759$,

so that $v^* = 40$ from (7).

<u>Case 2</u>: $f(v) = 50v^2.$

Equation (10) reduces to

$$22e^{0.1T^*} - e^{1.1T^*} = 21,$$

which yields $T^* = 1.236$.

Equation (9) reduces to

$$v^* = 0.909(1 - e^{-1.1T^*}),$$

yielding $v^* = 0.676$.

4. *Deterministic Undiscounted Models*

4.1 Finite Horizon Models:

Let T be the horizon length, and n the number of drillings; then the objective is to

$$\underset{v_i, t_i}{\text{Maximize}}\ J_n = \sum_{i=1}^{n} \left[\int_{t_i}^{t_{i+1}} PDv_i e^{-D(t-t_i)} dt - f(v_i) \right], \qquad (13)$$

where $t_1 = 0$ and $t_{n+1} = T$.

Equation (13) may be rewritten as

$$J_n = \sum_{i=1}^{n} \left[Pv_i(1 - e^{-D(t_{i+1}-t_i)}) - f(v_i) \right]. \qquad (14)$$

We need to analyze two cases, as before.

<u>Case 1</u>: $f''(v) \leq 0$.

As in Section 3, the concavity of $f(v)$ implies, from (14), that J_n is maximized by letting v_i become infinitely large; so we assume that $v_i \leq V < \infty$.

First suppose that only a single well may be drilled in the interval [0,T]. For the afore-mentioned reason, the optimal drilling capacity must be at the upper bound, provided that it yields a positive profit. Thus

$$v_1^* = \begin{cases} V & \text{if } PV(1-e^{-DT}) > f(V) \\ 0 & \text{otherwise} \end{cases}$$

and, of course, $t_1^* = 0$.

Now we assume that exactly n wells must be drilled in the interval [0,T]. Again, the concavity of $f(v)$ implies that

$$v_1^* = . \ . \ . \ . = v_n^* = V.$$

Equating to zero the derivatives of J_n with respect to t_i, $i = 1,\ldots,n$, we can show that

$$t_1^* = 0, \quad t_2^* = \frac{T}{n},\ldots,t_n^* = \frac{(n-1)T}{n} .$$

(The above is true subject to profitability).

The profit function is evaluated at the optimum as follows:

$$J_n^* = nPV(1 - e^{-DT/n}) - nf(V) . \tag{15}$$

The circumstances under which n wells are more profitable than $(n-1)$ wells may be determined by comparing J_n^* with J_{n-1}^*; the condition

$$J_n^* - J_{n-1}^* > 0 \tag{16}$$

reduces to

$$1 - \frac{f(V)}{PV} + (n-1)e^{-DT/(n-1)} - ne^{-DT/n} > 0 \quad (17)$$

in this case.

Define

$$Y_n = J_n^* - J_{n-1}^* \quad . \quad (18)$$

The following properties enable us to propose a method for determining the optimal number of wells n^*. Note that

$$\lim_{n\to\infty} J_n^* = -\infty ;$$

thus we may be certain that $n^* < \infty$.

Proposition 2: J_n^* is strictly concave in n, for $0 \le n < \infty$.

Proof: It is easily shown that

$$\frac{d^2 J_n^*}{dn^2} = - \frac{PVD^2T^2}{n^3} e^{-DT/n} < 0.$$

Q.E.D.

Proposition 3: Y_n is monotonically decreasing in n, for $1 \le n < \infty$.

Proof: The result is a direct consequence of the strict concavity of J_n^*.

Q.E.D.

Note that

$$Y_1 = J_1^* < \infty \ . \quad (19)$$

Proposition 3, in conjunction with (19), permits us to conclude that there exists some value of n, say n^*, satisfying

$$Y_{n^*} \geq 0,$$
$$Y_{n^*+1} < 0. \tag{20}$$

Then J_n^* attains a maximum at $n = n^*$, so that (20) represents the condition for optimality of n^*.

Condition (16) may also be viewed in a different light. Replacing the inequality with an equality, we may solve for T to obtain various "points of indifference" T_{n^*} for different values of n. Suppose T_1 is the solution for $n = 1$; then T_1 may be interpreted as the minimum horizon length for which $n^* = 1$ (or equivalently, the maximum horizon length for which $n^* = 0$). These points of indifference possess the following property:

<u>Proposition</u> 4: T_{n^*} is monotonically increasing in n^*, for $1 \leq n^*$.

<u>Proof</u>: We seek to show that $\frac{dT_{n^*}}{dn^*} > 0$. Replacing the inequality in (17) with an equality and letting $T = T_{n^*}$, $n = n^*$, we implicitly differentiate T_{n^*} with respect to n^* to obtain

$$\frac{dT_{n^*}}{dn^*} = \frac{\left(1+\frac{DT_{n^*}}{n^*}\right)e^{-DT_{n^*}/n^*} - \left(1+\frac{DT_{n^*}}{(n^*-1)}\right)e^{-DT_{n^*}/(n^*-1)}}{D\left(e^{-DT_{n^*}/n^*} - e^{-DT_{n^*}/(n^*-1)}\right)}.$$

Clearly the denominator > 0; so we need to show that the numerator > 0. Letting

$$Z_n = \left(1 + \frac{DT}{n}\right)e^{-DT/n},$$

we see that

$$\frac{dZ_n}{dn} = \frac{D^2T^2}{n^3}\, e^{-DT/n} > 0 \quad ,$$

completing the proof.

Q.E.D.

As a consequence of Proposition 4, the optimal number of wells n^* must satisfy

$$T_{n^*} \le T < T_{n^*+1} \quad . \tag{21}$$

Case 2: $f''(v) > 0$.

As in Section 3, the strict convexity of $f(v)$ implies that $v_i^* < \infty$, so that no upper bound is required on v_i.

Differentiating J_n with respect to t_i, v_i, $i = 1,\ldots,n$, we can show that

$$t_1^* = 0,\ t_2^* = \frac{T}{n},\ldots,t_n^* = \frac{(n-1)T}{n} ,$$

and $v_1^* = \ldots = v_n^* = V_n$, where V_n is the solution to

$$f'(V_n) = P(1 - e^{-DT/n}) \quad . \tag{22}$$

It is easily shown, as in Proposition 1, that $V_n > 0$.

Substituting for t_i^* and v_i^* in (14), we get

$$J_n^* = nPV_n(1 - e^{-DT/n}) - nf(V_n) \quad . \tag{23}$$

Defining Y_n as in (18), we may use (20) or (21) to determine n^*.

Example:

Suppose $T = 20$, $D = 1$, $P = 100$.

Case 1: $f(v) = 10v$, $V = 40$

We first evaluate

$$Y_n = 4000[0.9 + (n-1)e^{-20/(n-1)} - ne^{-20/n}],$$

and use condition (2) to obtain $n^* = 28$.

Then the drilling interval = 0.526, and $v_i^* = 40$.

Case 2: $f(v) = 10v^2$.

As before, we compute

$$Y_n = 250\ [n(1 - e^{-20/n})^2 - (n-1)(1 - e^{-20/(n-1)})^2]$$

and use condition (20) to obtain $n^* = 16$.

Then the drilling interval $= 1.25$, and $v_i^* = 3.567$.

4.2 Infinite Horizon Models:

It is clear by symmetry that the interval T between drillings will remain constant; so will the capacity v of the oil well. If K represents the average profit, the objective is to

$$\underset{v,T}{\text{Maximize}}\quad K = \frac{1}{T}\left[\int_0^T PDve^{-Dt}dt - f(v)\right]. \tag{24}$$

The above expression may be rewritten as

$$K = \frac{1}{T}\left[Pv(1 - e^{-DT}) - f(v)\right]. \tag{25}$$

We analyze the two cases as follows:

Case 1: $f'(v) \leq 0$.

As before, the concavity of $f(v)$ forces us to impose an upper bound on v, so we let $v \leq V < \infty$. Then the optimal oil well capacity is given by

$$v^* = \begin{cases} V & \text{if } PV(1 - e^{-DT^*}) > f(V) \\ 0 & \text{otherwise} \end{cases}. \tag{26}$$

The optimal drilling interval is determined by differentiating K with respect to T; thus T^* is the solution to

$$e^{-DT^*}(1 + DT^*) = 1 - \frac{f(V)}{PV}. \tag{27}$$

Case 2: $f'(v) > 0$.

As we have explained, no upper bound is required on v, as a consequence of the convexity of $f(v)$.

The optimal values are obtained by differentiating K with respect to v and T respectively; so we solve for v^* and T^* from the equations

$$f'(v^*) = P(1 - e^{-DT^*}) \tag{28}$$

and

$$T^* = \frac{v^* f'(v^*) - f(v^*)}{Dv^*(P - f'(v^*))} \quad . \tag{29}$$

It is easily shown, as in Proposition 1, that $v^* > 0$, so that it is always optimal to drill. Using (28), (29), in (25), the maximum average profit is given by

$$K^* = PDv^* e^{-DT^*} . \tag{30}$$

Note that K^* is equal to the average profit obtained by selling oil at the constant rate of $Dv^* e^{-DT^*}$, with no drilling costs.

5. Stochastic Models

It is often the case that the outcome of a drilling operation is, a priori, unknown; so we now treat the capacity v of the oil well as a random variable, instead of as a decision variable. Let $\varphi(v)$ be the probability density function, $\Phi(\cdot)$ the cumulative probability distribution, and $\bar{v}$ the expected value of the random variable v. We assume that a fixed cost Q is incurred per drilling. J now represents the expected profit function.

5.1 Finite Horizon Models:

Single-well Model:

Here we admit the possibility of drilling no more than a single well in the interval [0,T], where T is finite. The objective is to

$$\text{Maximize} \quad J_1 = E\left[\int_0^T PDve^{-Dt}dt - Q\right]$$

or equivalently,

$$J_1 = \bar{v}P(1 - e^{-DT}) - Q \;, \tag{31}$$

where $\bar{v}$ is the expected value of v. It is obvious, from (31), that the optimal policy is to drill (at $t_1^* = 0$) iff

$$\bar{v}P(1 - e^{-DT}) > Q. \tag{32}$$

Two-well Model:

We assume that no more than two wells may be drilled in the interval [0,T]. We first remark that consideration of the two-well case implies that it must be profitable to drill the first well; that is, (32) holds. The issue, therefore, reduces to deciding if, and when, a second well is to be drilled, contingent on the realization of the initial drilling, which we shall denote by w. We then seek to

$$\underset{t_2}{\text{Maximize}} \; J_2 = \int_0^{t_2} PDwe^{-Dt}dt + E\left[\int_{t_2}^{T} PDve^{-D(t-t_2)}dt\right] - 2Q$$

or equivalently,

$$J_2 = Pw(1 - e^{-Dt_2}) + P\bar{v}(1 - e^{-D(T-t_2)}) - 2Q, \tag{33}$$

where t_2 represents the time at which the second well is drilled, given that two wells must be drilled in the interval [0,T]. Differentiating (33) with respect

to t_2, we obtain the optimal solution

$$t_2^* = \frac{T}{2} - \frac{1}{2D} (\ln \bar{v} - \ln w) \ . \tag{34}$$

Further analysis is necessary to identify the best course of action depending on the outcome w of the first drilling. Notice that t_2^* is a monotonically increasing function of w. Let w_1 and w_2 represent the values of w for which $t_2^* = 0$ and $t_2^* = T$, respectively (see Fig. 3). Then from (34),

$$w_1 = \bar{v}\, e^{-DT} \tag{35}$$

and

$$w_2 = \bar{v}\, e^{DT} \ . \tag{36}$$

The plot of t_2^* as a function of w is shown below:

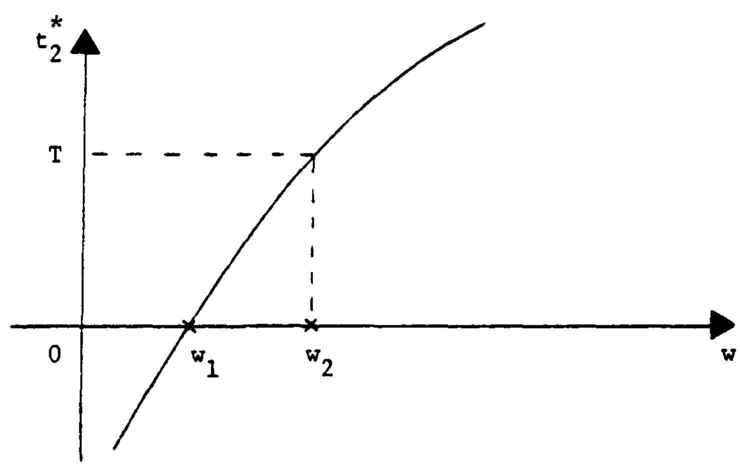


Figure 3

We now analyze several cases:

Case 1: $w > w_2$.

Here (34) and (36) together imply that $t_2^* > T$ (see Fig. 3); consequently, an optimal policy will not include a second well.

Case 2: $w_1 < w < w_2$.

It is easily inferred from (34), (35), (36) that $0 < t_2^* < T$ (see Fig. 3), so that a second well should be drilled at t_2^* if

$$J_2^* > J_1^* . \tag{35}$$

J_1^*, J_2^* are computed based on the realization of the initial drilling; thus (31) reduces to

$$J_1^* = wP(1 - e^{-DT}) - Q, \tag{36}$$

while (33) and (34) together yield

$$J_2^* = (w + \bar{v})P - 2\sqrt{w\bar{v}}\, e^{-DT/2} - 2Q. \tag{37}$$

Then condition (35) may be restated as follows: A second well should be drilled at t_2^* if

$$w < w_3 , \tag{37}$$

where

$$w_3 = (\sqrt{\bar{v}} - \sqrt{Q/P}\,)^2\, e^{DT}. \tag{38}$$

Case 3: $w < w_1$.

Equations (34) and (35) together imply that $t_2^* < 0$ (see Fig. 3); then the optimal policy is to abandon the first well and redrill immediately (at $t_2^* = 0$), provided this action enhances the expected profit.

Treating the cost incurred in the initial drilling as a sunk cost, the profit generated by retaining the first well throughout the interval [0,T] is $wP(1-e^{-DT})$. On the other hand, abandoning the first well and drilling anew at $t_2^* = 0$ yields an expected profit of $\bar{v}P(1 - e^{-DT}) - Q$. The latter course of action is desirable, therefore, if

$$\bar{v}\, P(1 - e^{-DT}) - Q > wP(1 - e^{-DT})$$

or equivalently, if

$$w < w_4, \tag{39}$$

where

$$w_4 = \bar{v} - \frac{Q}{P(1 - e^{-DT})} \quad . \tag{40}$$

Note that w_2 is larger than w_1, w_3, and w_4; the relative magnitudes of the latter three quantities, however, depend on Q, P, and T.

The analysis is summarized below:

1. $w > w_2$: do not drill.
2. $w_1 < w < w_2$: drill at t_2^* iff $w < w_3$,
   where t_2^* is given by (34).
3. $w < w_1$: drill at $t_2^* = 0$ iff $w < w_4$.

   The values of w_1, w_2, w_3, and w_4 are obtained from (35), (36), (38), and (40) respectively.

<u>Example</u>:

Let $T = 5$, $P = 100$, $Q = 1000$, $\bar{v} = 15$, $D = 1$.

First note that

$$\bar{v}\, P(1 - e^{-DT}) = 1489.89 > Q = 1000,$$

so that the condition for drilling the first well is satisfied.

Using (35), (36), (38), and (40), we get

$$w_1 = 0.101$$
$$w_2 = 2226.2$$
$$w_3 = 74.96$$
$$w_4 = 4.93 \ .$$

Notice that $w_1 < w_4 < w_3 < w_2$.

Since $w_3 > w_1$, decision rules 1 and 2 reduce to:

1. $w > 74.96$: do not redrill.
2. $0.101 < w < 74.96$: redrill at t_2^*.

Since $w_4 > w_1$, decision rule 3 reduces to:

3. $w < 0.101$: redrill immediately.

Suppose $w = 10$; then the optimal policy is to drill a second well at

$$t_2^* = 2.297 \quad \text{(from (34))}.$$

5.2 Infinite Horizon Model:

Assume a continuous discount rate $r > 0$.

The depletion of oil reserves in the well is accompanied by a decrease in the rate of oil extraction, so that a stage is reached when it becomes desirable to abandon the current well and drill a new one. The state of the process is completely determined by the oil reserve C. Thus there exists a level $\hat{C}$ such that the optimal policy assumes the following form:

Retain the current well if $C > \hat{C}$,

Drill a new well if $C \le \hat{C}$.

In view of the random nature of v, the above policy has the following implications: if $v \leq \hat{C}$, the well is abandoned and a new one drilled immediately; while if $v > \hat{C}$, the well is operated until $ve^{-Dt} = \hat{C}$, where t is the length of time for which the well has been in operation. We seek an expression for $\hat{C}$.

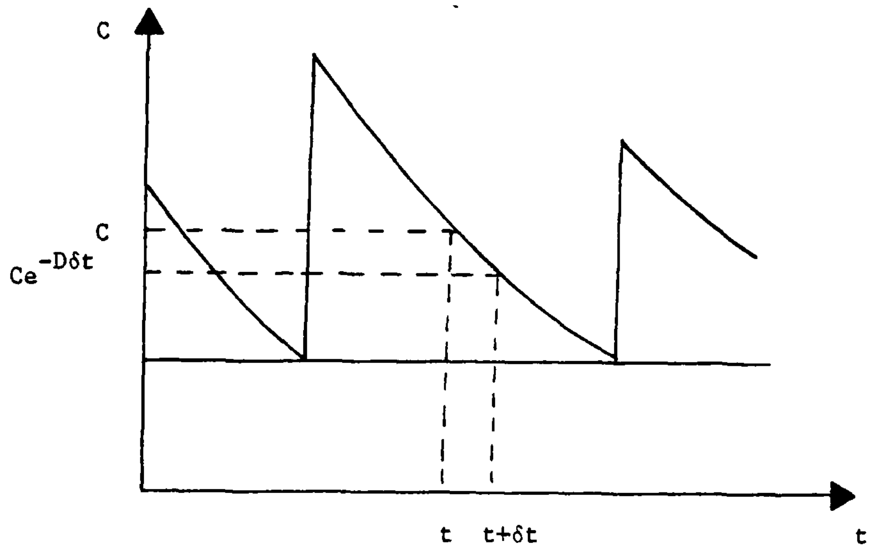


Figure 4

Let $J(C)$ be the value function representing the expected profit from t to infinity, given that the oil reserve at time t is C. It is easy to see that

$$J(C) = J(\hat{C}) \quad \text{for} \quad C \leq \hat{C}, \tag{41}$$

since the optimal policy involves redrilling whenever $C \leq \hat{C}$. On the other hand, suppose $C > \hat{C}$ (see Fig. 4). Then the quantity of oil in the well will decrease from C to $Ce^{-D\delta t}$ in the interval $[t, t + \delta t]$ (from (3)), generating a profit of $PC(1 - e^{-D\delta t})$. Thus we may write

$$J(C) = PC(1 - e^{-D\delta t}) + e^{-r\delta t} J(Ce^{-D\delta t}) . \tag{42}$$

Using the Taylor series expansion for e, writing

$$J(Ce^{-D\delta t}) \approx J(C - CD\delta t) \approx J(C) - J'(C)CD\delta t,$$

and omitting the square and higher powers of δt, (42) may be reduced to the following differential equation:

$$J'(C) + \frac{\rho J(C)}{C} = P, \tag{43}$$

where $\rho = r/D$.

Using C^{ρ} as the integrating factor, (43) may be solved to obtain

$$J(C) = \frac{P}{(\rho+1)} C + KC^{-\rho} , \tag{44}$$

where K is a constant of integration.

When C decreases to $\hat{C}$, the well is abandoned and a new one drilled. Conditioning on the outcome of the drilling, we may write

$$J(\hat{C}) = \text{Prob}(v>\hat{C})[J(v)-Q] + \text{Prob}(v\leq\hat{C})[J(\hat{C}) - Q] , \tag{45}$$

which simplifies to

$$J(\hat{C}) = \frac{1}{1 - \Phi(\hat{C})} [\int_{\hat{C}}^{\infty} \varphi(v)J(v)dv - Q] . \tag{46}$$

Note that (46) holds only for $\hat{C} > 0$; if $\hat{C} = 0$, then the optimal policy involves only a single drilling (at $t = 0$), so that

$$J(\hat{C}) = \int_0^{\infty} PD\bar{v}e^{-Dt}e^{-rt}dt - Q = \frac{P\bar{v}}{(1+\rho)} - Q, \quad \text{for } \hat{C} = 0.$$

If the above expression is negative, that is, if

$$\frac{P\bar{v}}{(1+\rho)} < Q \,, \tag{47}$$

then it will never be optimal to drill.

Our objective here is to maximize the value function at $t = 0$. Assuming that there is no functioning oil well at the beginning of the process, it follows from (41) that the objective is to

$$\underset{\hat{C}}{\text{Maximize}} \quad J(\hat{C}) \,,$$

where $J(\hat{C})$ is given by (46). To evaluate (46), we substitute for $J(v)$ from (44); but first we need an expression for the constant of integration K. As a boundary condition, we require that $J(C)$, evaluated at $\hat{C}$ from (44), equals $J(\hat{C})$ from (46), so that $J(C)$ is continuous at $\hat{C}$. We now demonstrate intuitively that the boundary condition we have just imposed is entirely appropriate.

Suppose $J(C)$ takes a downward jump at $\hat{C}$ (see Fig. 5). Then we are clearly better off by increasing $\hat{C}$, and we can keep doing this until the two parts of $J(C)$ intersect.

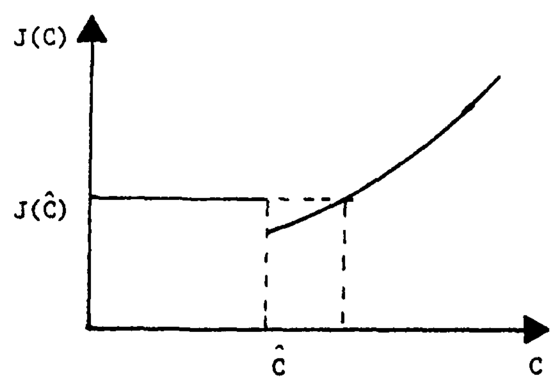


Figure 5

On the other hand, suppose $J(C)$ takes an upward jump at $\hat{C}$ (see Fig. 6). Then we can increase profits by decreasing $\hat{C}$, and this process continues until the two parts of $J(C)$ intersect.

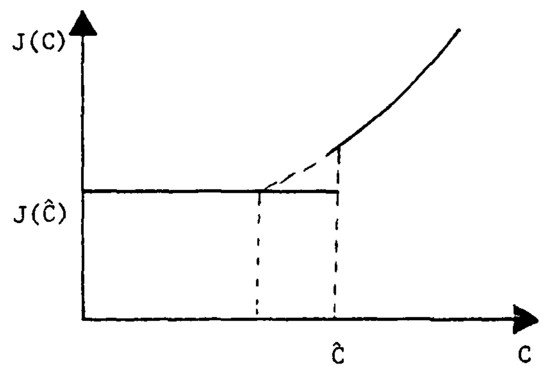


Figure 6

As a consequence of the assumed boundary condition, we have

$$J(\hat{C}) = \frac{P}{(\rho+1)}\hat{C} + K\hat{C}^{-\rho} = \frac{1}{1-\Phi(\hat{C})}\left[\int_{\hat{C}}^{\infty}\varphi(v)\left(\frac{P}{(\rho+1)}v+Kv^{-\rho}\right)dv - Q\right],$$

from which we solve for K as a function of $\hat{C}$:

$$K(\hat{C}) = \frac{\frac{P}{(\rho+1)}\int_{\hat{C}}^{\infty}\varphi(v)(v-\hat{C})dv - Q}{\int_{\hat{C}}^{\infty}\varphi(v)(\hat{C}^{-\rho}-v^{-\rho})\,dv} \qquad (48)$$

It is easily verified from (48) that $K(0) = 0$, and $\lim_{\hat{C}\to\infty} K(\hat{C}) = -\infty$.

Now $K(\hat{C})$ merely represents the value of the constant of integration K which equates $J(C)$, evaluated at $\hat{C}$ from (44), with $J(\hat{C})$ from (46). Further,

$J(C)$ is increasing in K (from (44)). Thus, if $J(\hat{C})$ attains a maximimum at $\hat{C}^*$, it is clear that $K(\hat{C})$ must increase with respect to $\hat{C}$ in $[0,\hat{C}^*]$, and decrease thereafter; in other words, $K(\hat{C})$ also attains a maximum at $\hat{C}^*$ (see Fig. 7).

Since the objective is to maximize $J(\hat{C})$, or equivalently, $K(\hat{C})$, with respect to $\hat{C}$, we differentiate (48) with respect to $\hat{C}$ and equate it to zero; accordingly, $\hat{C}^*$ is obtained as the solution to

$$\int_{\hat{C}^*}^{\infty} \varphi(v)(v-\hat{C}^*)dv - \frac{\hat{C}^*(1+\rho)}{\rho} \int_{\hat{C}^*}^{\infty} \varphi(v)(\hat{C}^{*-\rho} - v^{-\rho})dv = \frac{Q(\rho+1)}{P} \qquad (49)$$

Differentiating (48) twice with respect to $\hat{C}$, and using (49), we may show that

$$K''(\hat{C}^*) = -\frac{PD}{\hat{C}^* \int_{\hat{C}^*}^{\infty} \varphi(v)(\hat{C}^{*-\rho} - v^{-\rho})dv} < 0 ,$$

so that the condition for a maximum is satisfied.

$K(\hat{C}^*)$ is evaluated using (48) and (49):

$$K(\hat{C}^*) = \frac{P\hat{C}^*(1+\rho)}{\rho(1+\rho)} \quad . \qquad (50)$$

The value of the objective function at the optimum is obtained from (44) and (50):

$$J(\hat{C}^*) = \frac{P\hat{C}^*}{\rho} \quad . \qquad (51)$$

Equation (51) affords an interesting interpretation. Note that

$$\int_0^{\infty} PD\hat{C}^* e^{-rt} dt = \frac{P\hat{C}^*}{\rho} \quad .$$

Thus the optimal choice of $\hat{C}$ for the original problem yields an infinite horizon discounted profit which is exactly equal to that derived from a constant supply of oil at the rate $D\hat{C}^*$, with no drilling costs; further, this constant rate is precisely the rate of oil extraction from the well at the instant prior to abandonment.

We show diagrammatically the relationship between $J(C)$, $J(\hat{C})$, and $K(\hat{C})$ in Figure 7:

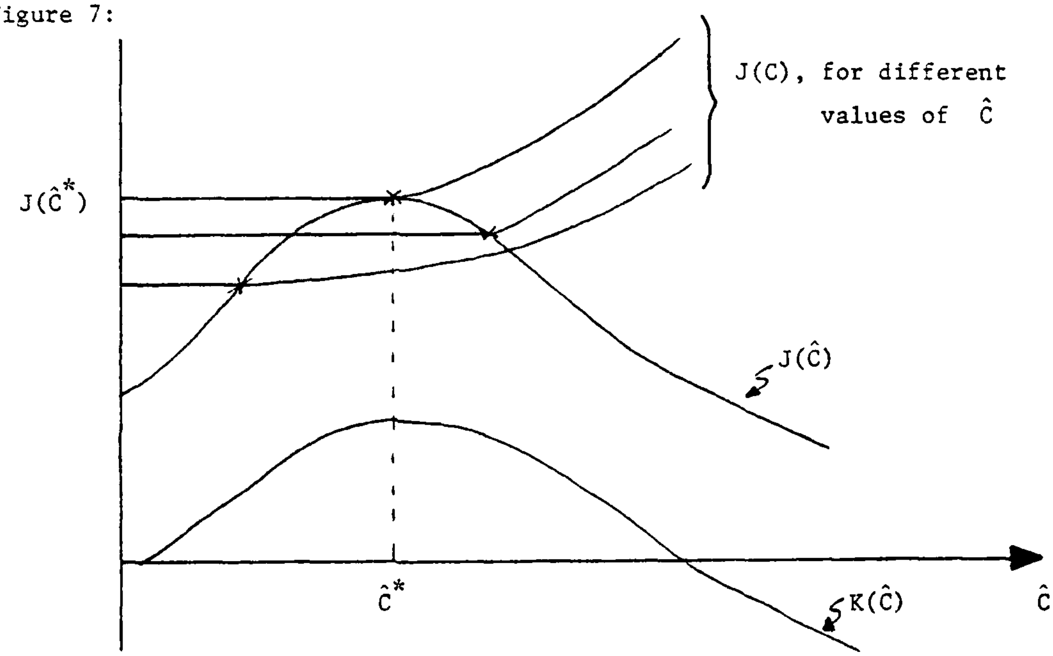


Figure 7

Finally, we study the effect of variation of different parameters on $\hat{C}^*$. Representing the left hand side of (49) by F, we can show that

$$\frac{\partial F}{\partial Q} = \frac{(\rho+1)}{P} > 0 \quad , \tag{52}$$

$$\frac{\partial F}{\partial P} = -\frac{Q(\rho+1)}{P^2} < 0, \tag{53}$$

$$\frac{\partial F}{\partial \rho} = \frac{Q}{P} > 0. \tag{54}$$

Further,

$$\frac{\partial F}{\partial \hat{C}^*} = -\frac{(1+\rho)}{\rho} \hat{C}^{*\rho} \int_{\hat{C}^*}^{\infty} \varphi(v)(\hat{C}^{*-\rho} - v^{-\rho})dv < 0. \tag{55}$$

From (52) and (55), we have

$$\frac{\partial \hat{C}^*}{\partial Q} < 0 \quad , \tag{56}$$

while (53) and (55) yield

$$\frac{\partial \hat{C}^*}{\partial P} > 0 \quad . \tag{57}$$

By the implicit function theorem, we have

$$\frac{\partial \hat{C}^*}{\partial \rho} = -\frac{\partial F/\partial \rho}{\partial F/\partial \hat{C}^*} \quad ,$$

which, in conjunction with (54) and (55), yields

$$\frac{\partial \hat{C}^*}{\partial \rho} > 0. \tag{58}$$

From (56), (57), and (58), we conclude that the optimal redrilling level $\hat{C}^*$ decreases with the drilling cost Q and the decay constant D, while it increases with the unit profit P and the discount rate r.

<u>Example:</u>

We evaluate (49) for the uniform density on [0,1], thus:

$$\frac{\hat{C}^*[\hat{C}^{*\rho} - \rho\hat{C}^* + \rho - 1]}{\rho(1-\rho)} + \frac{1}{2}(1-\hat{C}^*)^2 = \frac{Q(1+\rho)}{P} .$$

For $P = 1000$, $Q = 100$, $r = 0.1$, $D = 1$, the above equation reduces to

$$\frac{\hat{C}^*[\hat{C}^{*0.1} - 0.1\hat{C}^* - 0.9]}{0.09} + \frac{1}{2}(1 - \hat{C}^*)^2 = 0.11,$$

which may be solved to yield $\hat{C}^* = 0.282$.

6. Concluding Remarks

The models developed in this paper appear to permit a fairly wide range of practical applications. Situations calling for slightly different sets of assumptions could easily be analyzed in similar fashion.

References

[1] Bensoussan, A., Hurst, E.G. and Näslund, B., Management Applications of Modern Control Theory, North-Holland, Amsterdam (1974).

[2] Blaquière, A., "Necessary and sufficiency conditions for optimal strategies in impulsive control," presented at the Third Kingston Concerence on Differential Games and Control Theory, University of Rhode Island (1978).

[3] Case, J., "Impulsively Controlled Differential Games," in The Theory and Application of Differential Games (J. D. Grote, Ed.), D. Reidel Publishing Company, Dordrecht (1975).

[4] Clarke, D.G., "Econometric Measurement of the Duration of the Advertising Effect on Sales," Journal of Marketing Research, Vol. 13, No. 4, (1976), pp.423-40.

[5] Pekelman, D. and Sethi, S.P., "Advertising Budgeting, Wearout, and Copy Replacement," Journal of Op. Res. Soc., Vol. 29, No. 3, (1978), pp. 651-659.

[6] Rapp, B., Models for Optimal Investment and Maintenance Decisions, Halsted Press, Stockholm (1974).

[7] Sethi, S.P. and Thompson, G.L., Applied Optimal Control and Management Science, Martinus Nijhoff, Boston, (1981).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: M.S.R.R. 464
2. GOVT ACCESSION NO.: AD-A092 522
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): PROFIT MAXIMIZATION MODELS FOR EXPONENTIAL DECAY PROCESSES
5. TYPE OF REPORT & PERIOD COVERED: Technical Report, August 1980
6. PERFORMING ORG. REPORT NUMBER: M.S.R.R. 464
7. AUTHOR(s): S. P. Sethi, G. L. Thompson, V. Udayabhanu
8. CONTRACT OR GRANT NUMBER(s): N00014-75-C-0621, A4619 NSERC, Canada
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NR 047-048
11. CONTROLLING OFFICE NAME AND ADDRESS: Personnel and Training Research Programs, Office of Naval Research (Code 458), Arlington, Virginia 22217
12. REPORT DATE: August 1980
13. NUMBER OF PAGES: 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Exponential decay processes, Oil driller's problem
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

A number of real workd processes can be modelled as exponential decay processes. Examples are: machine replacement, oil well extraction, advertising goodwill, repair and cleaning activities, etc.. In this paper we analyze a series of discounted or undiscounted, determinisitic or stochastic exponential decay models. We characterize finite and infinite horizon optimal solutions for each model. We show that the solution can be characterized for the oil drillers problem in the following way: Once a well of sufficient

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE — S/N 0102-014-6601

(OVER)

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

capacity is drilled, oil is pumped from it until the oil remaining decreases to a fixed cut-off level; then the well is abandoned, and a new well is drilled. The resulting process when repeated over time appears to be the same as an oil source which produces oil revenue continuously at the fixed cut-off level. In other words, the excess revenue received from an oil well when its capacity is greater than the cutoff level is just sufficient to pay for drilling costs for a new well.